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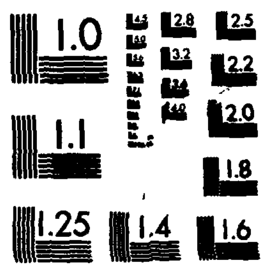

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ARL-STRUC-R-421

AR-004-493

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MELBOURNE, VICTORIA

Structures Report 421

ON THE APPLICATION OF THE STRAIN ENERGY DENSITY THEORY IN PREDICTING CRACK INITIATION AND ANGLE OF GROWTH

by

A.K. WONG

Approved for Public Release

SEPTEMBER 1986

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SUMMARY

Until recently, the one-parameter singular expression for stresses near a crack-tip was widely thought to be sufficiently accurate over a reasonable region for any geometry and loading conditions. This view has been fast changing due to the growing evidence that the inclusion of higher order terms can significantly affect the solution, particularly for certain biaxial loading conditions. In this context, the present paper examines the strain energy density criterion for fracture, and the consequences of the assumption of a 1/r energy singularity in the formulation on its application. It is found that this assumption imposes a rather severe restriction on the region for which the criterion is applicable, and that its application on an arbitrarily selected 'small' distance from the crack-tip (a procedure which has been adopted by many experimentalists and finite element analysts), can lead to erroneous results.

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NOTATION

A, B, C	First three coefficients of the series expansion of the strain energy density function
a	Half crack length of specimen
b	Half width of specimen
E	Elastic modulus
F	Function of elastic modulus and Poisson's ratio
h	Half height of specimen
i	$\sqrt{-1}$
J	J-integral
J_c	Critical J for determining fracture
K_I, K_{II}	Mode-I and Mode-II Stress intensity factors
K_{I_c}	Critical stress intensity factor for Mode-I fracture
k	Ratio of remote lateral load to remote vertical load
r	Radial distance from crack-tip
r_j	Incremental crack growth
r_c	Critical crack growth increment
r_o	Radius defining core region
S	Strain energy density factor
S_o, S_1, S_2	First three coefficients of the series expansion of the strain energy density factor
S_j	Strain energy density factor at load increment j
S_c	Critical strain energy density factor for determining fracture
s	Integration path for the J-integral
$\mathbf{T}$	Traction vector on s
$\mathbf{u}$	Displacement vector on s
v_1	Displacement in the y-direction at the first corner node behind the crack-tip
W	Strain energy density function
W_c	Critical strain energy density function
x, y	Cartesian coordinate axis system
z	Complex vector $x + iy$
α	Length parameter $\sqrt{2r/a}$
β	Inclination of the crack
Γ, Γ'	Complex function of load, biaxiality and angle of inclination
Δ_1	Distance of the first corner node behind the crack-tip to the crack-tip
$\epsilon_x, \epsilon_y, \epsilon_{xy}$	Normal and shear strains
θ	Angular coordinate at crack-tip
λ	Function of Poisson's ratio
ν	Poisson's ratio
π	Ratio of the circumference to the diameter of a circle
$\sigma_x, \sigma_y, \sigma_{xy}$	Normal and shear stresses
$\Phi, \Omega, \phi', \omega'$	Holomorphic functions

1. INTRODUCTION

The analysis of the stress field around cracks in an elastic solid has been well documented and is perhaps much older than the field now known as Linear Elastic Fracture Mechanics. One of the early works is due to Westergaard [1], who treated the sharp crack problem using an eigenfunction expansion approach. In a later paper, Westergaard [2] expressed the solution to the same problem in terms of complex analytic functions. This later work has since gained much recognition, and is frequently referenced by the analysts of fracture mechanics. It can be shown that the Westergaard solution leads to the following classical result for the stress field near the tips of a centre-crack contained in an infinite plate loaded by a remote uniaxial tensile stress σ (see Fig. 1):

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right), \tag{1.1a}$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right), \tag{1.1b}$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}. \tag{1.1c}$$

where K_I is known as the stress intensity factor.

However, a small error made in the Westergaard solution remained undetected for almost thirty years. By using the more general complex potentials approach of Muskhelishvili [3], Sih [4] showed that the arbitrary setting of a particular constant in the Westergaard solution to zero was generally invalid. Discussion on the effects of the error was subsequently taken up by Eftis and Liebowitz [5], and later, Eftis *et al* [6] showed that the error was equivalent to the omission of a non-singular term from the stress expressions. By considering an infinite centre-cracked plate under biaxial remote loads σ (perpendicular to the crack) and $k\sigma$ (parallel to the crack), it was shown that the solution, correct to the zero-th order term in r, is

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) - (1-k)\sigma, \tag{1.2a}$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right), \tag{1.2b}$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}. \tag{1.2c}$$

Comparing Eqs 1.2 with Eqs 1.1, and setting $k = 0$ for uniaxial loading, it is seen that the only difference is the non-singular term σ in the expression for σ_x. Of course, as $r \to 0$, this term is expected to become negligible. However, it does not do so as rapidly as one would like, mainly because of the square root effect of the singular term. For the uniaxial case as an example, $k = 0$ and $K_I = \sigma\sqrt{\pi a}$, so that for $\theta = 0°$, Eq. 1.2a becomes

$$\sigma_x = \sigma\left[\sqrt{\frac{a}{2r}} - 1\right]. \tag{1.3}$$

For the error caused by neglecting the non-singular term to be less than 1% say, it is required that

$$\sqrt{\frac{a}{2r}} \geq 100, \tag{1.4}$$

or

$$\frac{r}{a} \leq 5 \times 10^{-6}. \tag{1.5}$$

This represents an extremely small region around the crack-tip where the non-singular term may be regarded as being 'higher order'. Note also that as k becomes greater ($k \rightarrow \pm\infty$), the restriction on r/a would become even more severe. Eftis *et al* [6] showed that for the case of biaxial loading, the omission of the non-singular term in the prediction of isochromatic fringe patterns for photoelastic analyses, and the prediction of crack angles using the maximum normal stress criterion, leads to significant discrepancies with experimental data. In a subsequent paper, Eftis *et al* [7] demonstrated that the local elastic strain energy density and strain energy rate also depend significantly on the biaxiality of the applied load, and the arbitrary omission of the non-singular term denies the solution to the biaxial effects and therefore leads to incorrect predictions of the angle of growth when using Sih's strain energy density criterion [8].

This work continues the investigation on the effect of the omission of the non-singular term in the formulation of the S-theory, and the consequences and difficulties in its application as a result. It will be shown that even the simple uniaxial results may be significantly affected, and the unaware practitioner of the S-theory can be misled.

2. SOLUTION FOR THE INCLINED CRACK PROBLEM

To illustrate the effect of the omission of the non-singular term, we will turn to a problem for which a well known solution exists. Consider the infinite plate containing a central crack of length $2a$ and subjected to biaxial loads σ and $k\sigma$ as shown in Fig. 2. Following Muskhelishvili [9], the holomorphic functions which represent the solution are

$$\Phi(z) = \phi'(z) = \frac{1}{2}(2\Gamma + \Gamma')\frac{z}{\sqrt{z^2 - a^2}} - \frac{1}{2}\Gamma', \tag{2.1}$$

$$\Omega(z) = \omega'(z) = \frac{1}{2}(2\Gamma + \Gamma')\frac{z}{\sqrt{z^2 - a^2}} + \frac{1}{2}\Gamma', \tag{2.2}$$

where

$$\frac{1}{2}\Gamma' = -\frac{\sigma}{4}(1 - k)e^{2i\beta}, \tag{2.3}$$

$$\frac{1}{2}(2\Gamma + \Gamma') = \frac{\sigma}{4}\left\{(1 - e^{2i\beta}) + k(1 + e^{2i\beta})\right\}, \tag{2.4}$$

and the stresses $\sigma_x, \sigma_y, \sigma_{xy}$ are given by

$$\sigma_x + \sigma_y = 2(\Phi(z) + \overline{\Phi(z)}), \tag{2.5}$$

$$\sigma_y - i\sigma_{xy} = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z})\overline{\Phi'(z)}. \tag{2.6}$$

After some manipulation, and restricting the solution region to $0 < r/a \ll 1$, it may be shown that a series expansion of the solution gives

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) - \frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right) + \sigma(1-k)\cos 2\beta, \tag{2.7a}$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}, \tag{2.7b}$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} + \frac{K_{II}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right), \tag{2.7c}$$

where K_I and K_{II} are the stress intensity factors given by

$$K_I = \frac{\sigma\sqrt{\pi a}}{2}\left[(1+k)-(1-k)\cos 2\beta\right], \tag{2.8}$$

$$K_{II} = \frac{\sigma\sqrt{\pi a}}{2}(1-k)\sin 2\beta. \tag{2.9}$$

Following Hooke's law, the in-plane elastic strains are

$$\begin{aligned} \epsilon_x &= \frac{1}{F}(\sigma_x - \lambda\sigma_y), \\ \epsilon_y &= \frac{1}{F}(\sigma_y - \lambda\sigma_x), \\ \epsilon_{xy} &= \frac{\sigma_{xy}(1+\lambda)}{F}, \end{aligned} \tag{2.10}$$

where

$$F = E \qquad \lambda = \nu \qquad \text{for plane stress,}$$
$$F = \frac{E}{1-\nu^2}, \qquad \lambda = \frac{\nu}{1-\nu} \qquad \text{for plane strain,}$$

and E and ν are the elastic modulus and Poisson's ratio respectively.

Hence, the strain energy density function W for a plane problem is given by

$$\begin{aligned} W &= \frac{1}{2}(\sigma_x\epsilon_x + \sigma_y\epsilon_y + 2\sigma_{xy}\epsilon_{xy}) \\ &= \frac{1}{2F}\left[\sigma_x^2 + \sigma_y^2 + 2\sigma_{xy}^2 + 2\lambda(\sigma_{xy}^2 - \sigma_x\sigma_y)\right]. \end{aligned} \tag{2.11}$$

Substituting Eqs 2.7 into Eq. 2.11, we obtain

$$W = \frac{A}{r} + \frac{B}{\sqrt{r}} + C, \tag{2.12}$$

where

$$\begin{aligned} A = \frac{1}{16F\pi}\{&2K_I^2(1+\cos\theta)[(3-\cos\theta)-\lambda(1+\cos\theta)] \\ &+ K_{II}^2[(9-4\cos\theta+3\cos 2\theta)+\lambda(1+4\cos\theta+3\cos 2\theta)] \\ &+ 8K_IK_{II}\sin\theta[(\cos\theta-1)+\lambda(\cos\theta+1)]\}, \end{aligned}$$

$$\begin{aligned} B = \frac{\sigma(1-k)\cos 2\beta}{2F\sqrt{2\pi}}\{&K_I\cos\frac{\theta}{2}[(2-\cos\theta+\cos 2\theta)-\lambda(2+\cos\theta-\cos 2\theta)] \\ &- K_{II}\sin\frac{\theta}{2}[(4+\cos\theta+\cos 2\theta)+\lambda(\cos\theta+\cos 2\theta)]\}, \end{aligned}$$

$$C = \frac{1}{2F}\sigma^2(1-k)^2\cos^2 2\beta.$$

Note that the inclusion of the non-singular term in the stress expression leads to two additional terms in W, one of which has a $r^{-1/2}$ singularity whilst the other is non-singular. To see how this affects the application of the S-theory, it is instructive to summarise the basis of the theory.

2.1 The Strain Energy Density Criterion

The strain energy density criterion was first proposed by Sih [8], and its application has been well documented (see for example [9] – [11]). Its formulation is based on the assumption that the strain energy density function near the crack-tip possesses a $1/r$ singularity. This may be seen from Eq. 2.12 in which for sufficiently small r, W may be written as

$$W = \frac{A}{r}, \tag{2.13}$$

Hence, an r-independent parameter, known as the strain energy density factor S, may be defined such that

$$S = rW = A. \tag{2.14}$$

The criterion concerning crack initiation and the angle of growth is then based on the following three hypotheses:

Hypothesis 1: Crack extension begins along the direction where the strain energy density factor is a local minimum (denoted by S_{min}).

Hypothesis 2: Fracture is imminent when the local minimum S_{min} reaches a critical value S_c, and that S_c is a material parameter which characterises the fracture strength of the material.

Hypothesis 3: The amount of stable incremental growth $r_1, r_2, \ldots, r_j, \ldots, r_c$ is governed by

$$W_c = \frac{S_1}{r_1} = \frac{S_2}{r_2} = \ldots = \frac{S_j}{r_j} = \ldots = \frac{S_c}{r_c} = const. \tag{2.15}$$

where upon reaching r_c, unstable fracture occurs.

For a centre-cracked specimen under mode I conditions, both the direction of crack propagation and fracture toughness are well known. Hence, the substitution of $\theta = 0°, K_I = K_{I_c}, K_{II} = 0$ into Eq. 2.14 gives, for plane strain conditions, a convenient expression for relating S_c to existing fracture toughness data, viz.,

$$S_c = \frac{(1+\nu)(1-2\nu)}{2\pi E} K_{I_c}^2. \tag{2.16}$$

Figure 3 shows a graphical representation of Hypothesis 3. It shows that the amount of incremental growth can be predicted by the intersection of the plots of W and the material constant W_c. In an elastic analysis, W is unbounded as $r \to 0$ for any given applied stress σ so that W will always intersect W_c. In practice however, a crack will not grow unless a sufficient load is applied. To account for this phenomenon, a threshold distance r_o is defined such that for $r < r_o$ (referred to as the core region), continuum mechanics is assumed to fall short of reality so that any analysis, and hence prediction of growth, must be kept outside the core region. That is, the condition $r_j \geq r_o$ must be satisfied. It is also clear that for a material which exhibits negligible stable growth (i.e., a brittle material), $r_o = r_c$. For a 4140 steel with various yield strengths and fracture toughnesses, Sih [12] showed that r_o ranged from 0.00065 in. (0.0165 mm) to 0.01345 in. (0.3416 mm).

3. THE APPLICATION OF THE S-THEORY IN PREDICTING BRITTLE FAILURE

One advantage of the strain energy density criterion over most other fracture criteria is that both the angle of growth and critical load may be predicted by a single parameter. Furthermore, without the restriction of self-similar growth as required by other criteria such as the critical energy release rate approach, the S criterion may be applied in complex load systems. For example, by means of finite element modelling, the S distribution around a crack-tip contained in a complex structure under an arbitrary load system may, in theory, be used to predict the failure load as well as the direction of propagation. However, it will be shown in this section that without sufficient care, this practice can lead to erroneous results.

3.1 Predicting the Onset of Failure

Consider the uniaxial problem described previously. Since the direction of growth for this case is known *a priori* (namely along the x-axis), substituting $\theta = 0°, \beta = 90°, K_{II} = 0$ into Eq. 2.12 and multiplying through by r yields an expression for S along the critical direction, viz.,

$$S = \frac{1}{2F}\left[\frac{K_I^2}{\pi}(1-\lambda) - \sigma K_I(1-\lambda)\sqrt{\frac{2r}{\pi}} + \sigma^2 r\right]. \tag{3.1}$$

And since $K_I = \sigma\sqrt{\pi a}$,

$$\begin{aligned} S &= \frac{\sigma^2}{2F}[a(1-\lambda) - (1-\lambda)\sqrt{2ar} + r] \\ &= \frac{\sigma^2 a}{2F}\left[(1-\lambda) - (1-\lambda)\sqrt{2}\sqrt{\frac{r}{a}} + \left(\frac{r}{a}\right)\right]. \end{aligned} \tag{3.2}$$

The one-term representation, Eq. 2.14, may be obtained by taking the limit of Eq. 3.2 as $r/a \to 0$, giving

$$S = \frac{\sigma^2 a}{2F}(1-\lambda). \tag{3.3}$$

Taking a typical value of $\nu = 0.3$, Fig. 4 shows a comparison between the one-term representation and the higher order expression for S . It may be seen that there is significant difference between Eq. 3.2 and Eq. 3.3. At the relatively small value of $r/a = 0.02$, the neglect of the higher order terms gives rise to errors of approximately 20% in both the plane stress and plane strain results. Indeed, values of $r/a \geq 0.02$ have not been uncommon in many analyses using the strain energy density method (e.g., [10], [13]). What is more important in this example is that along the critical direction, the one-term expression always over-estimates the correct value of S for $r/a > 0$. Since S_c is derived from the one-term expression (Eq. 2.16), the prediction of fracture load using the S-theory by means of finite element modelling, or by physically monitoring the elastic strains at some small but finite distance from the crack tip, will invariably produce an over-estimate. In the above example, if the strain-energy density S at $r/a = 0.02$ is monitored, and Hypothesis 2 of the criterion is applied so that fracture load is assumed to be reached when $S = S_c$, then $S(r/a \to 0)$ would be greater than S_c by about 20% and hence the fracture strength would be over-estimated by 9.5%. The non-conservative nature and the relatively large magnitude of the error involved is certainly undesirable, and it shows how the blind application of the S-theory in design work can be extremely dangerous.

For the S-theory to be applicable, the analysis must be confined to a region for which the one-term expression, Eq. 3.3, is valid. Allowing an error of say 5 %, we get from Eq. 3.2

$$\left|\frac{(1-\lambda)\sqrt{2r/a}-r/a}{(1-\lambda)-(1-\lambda)\sqrt{2r/a}+r/a}\right| \leq 0.05. \tag{3.4}$$

The solution which satisfies the condition $0 < r/a \ll 1$ is given by

$$\frac{r}{a} \leq 0.00124. \tag{3.5}$$

Equation 3.5 represents an upper limit to the region where the strain energy density analysis may be applied with acceptable accuracy. However, we recall that there exists also a lower limit r_o which defines the core region (see §2.1). Consequently, the region where the S-theory is valid may be given by

$$r_o \leq r \leq 0.00124\,a. \tag{3.6}$$

Equation 3.6 implies

$$a \geq \frac{r_o}{0.00124}. \tag{3.7}$$

The restrictions inferred by Eqs 3.5 and 3.7 are severe, and in some cases, may be considered as impractical. For the high fracture strength 4140 steel ($r_o = 0.01345$ in.) quoted in [12] as an example, the minimum crack length required for a valid LEFM analysis would be 10.85 in. (275.6 mm). Such a condition would be difficult to satisfy, particularly for real life situations where cracks of much shorter lengths are detected and require analysis. Equation 3.5 on the other hand can almost always be satisfied in a finite element model in which mesh sizes of the order of $0.001\,a$ may be readily handled on a modern computer. However, using such a fine mesh for a linear analysis may be difficult to justify as stress intensity factors have been successfully computed by other methods with much coarser grids. To highlight this point, a finite element model of a centre-cracked plate ($\nu = 0.3$) subjected to uniaxial tensile loading and plane strain conditions was established. Because of symmetry, only one quarter of the plate which has a width to height ratio $b/h = 1$, and a crack-length to width ratio $a/b = 0.25$ was modelled. Figure 5 shows the two mesh schemes adopted for the analysis. The parameters calculated include

1) The stress intensity factor K_I given by the expression [14]

$$K_I = \frac{v_1 F}{2}\sqrt{\frac{\pi}{2\Delta_1}}, \tag{3.8}$$

where v_1 and Δ_1 are respectively the y-displacement and distance from the crack tip at the first corner node behind the crack tip, and F is as defined in Eq. 2.10.

2) The J-integral given by the numerical integration of the expression [15]

$$J = \int W\,dy - \mathbf{T}\cdot\frac{\partial \mathbf{u}}{\partial x}\,ds, \tag{3.9}$$

where **T** and **u** are the traction force and displacement vectors along the path s respectively. The integration path chosen in this case is shown in Fig. 5. It may be shown that J is path-independent for an elastic material and is related to the stress intensity factor by the following expression

$$K_I = \sqrt{JF}. \tag{3.10}$$

3) The strain energy density factor $S(= rW)$ at nodal points along the critical direction ($\theta = 0°$), where W is calculated from the computed stresses (Eq. 2.11). To estimate the stress intensity factor from the strain energy density factor, the conventional expression which neglects the higher order terms is used [12]

$$K_I = \sqrt{\frac{2\pi ES}{(1+\nu)(1-2\nu)}}. \tag{3.11}$$

In the following presentation, all values of K_I and S have been normalised by $\sigma\sqrt{a}$ and $\sigma^2 a/E$ respectively. The computed strain energy density along the critical direction ($\theta = 0°$) for Mesh-1 is shown in Fig. 6. An immediate observation is the systematic scatter of the results about some mean value. The strain energy density factor appears to be over-estimated at the corner nodes, and under-estimated at mid-side nodes. Whilst the average deviation of S from the mean amounts to approximately 5.7 %, the scatter of the computed stresses would be only approximately 2.8 % and is therefore considered acceptable. If the assumption of a constant S is made over the region say $0 < r/a \leq 0.04$, then a best fit to the data gives the result $S = 0.2597$, which when used in Eq. 3.11, gives $K_I = 1.771$. This compares with $K_I = 1.950$ obtained using the analytical solution given in Rooke and Cartwright [16], and therefore represents a 9.2 % under-estimation.

The computational results for the refined mesh are shown in Fig. 7. Again, a similar systematic scatter is apparent, and the assumption of a constant S in the range $0 < r/a \leq 0.04$ gives $S = 0.2827$, or from Eq. 3.11, $K_I = 1.848$ which is in error by 5.2 %. However, the downward trend of the S-distribution as predicted by the analytical solution is clearly evident in this case, and compares well with the plot of Eq. 3.1 (where K_I is taken to be 1.950). This suggests that some sort of curve fitting to the correct form

$$S = S_o + S_1\sqrt{\frac{r}{a}} + S_2\left(\frac{r}{a}\right), \tag{3.12}$$

would provide a more accurate prediction of strain energy singularity. This was indeed found to be the case. Applying a least squares fit for the S data of Mesh-2 to Eq. 3.12, it was found that $S_o = 0.3082$. Substituting S_o for S in Eq. 3.11 yields $K_I = 1.930$, giving an error of only 1 %.

Table 1 summarises the various methods of computing the stress intensity factor and their errors with respect to the analytical solution obtained in [16].

K_I Ref. [16]	$K_I(v_1, \Delta_1)$		$K_I(J)$		$K_I(S = const.)$		$K_I(S_o)$
	Mesh-1	Mesh-2	Mesh-1	Mesh-2	Mesh-1	Mesh-2	Mesh-2
1.950	1.892	1.902	1.894	1.896	1.771	1.848	1.930
% Error	3.0	2.5	2.9	2.8	9.2	5.2	1.0

Table 1. Stress Intensity Factors Calculated by Various Methods

It may be seen from Table 1 that the stress intensity factor is predicted accurately by both the crack opening and the J-integral even for Mesh-1. The evaluation of K_I from S however required the much finer mesh to achieve an accuracy to within 5 %. For a more accurate prediction, a least squares fit of the correct form was required. Whilst this curve fitting technique proved to be useful in the case of Mesh-2, it would be unreasonable to apply this to the data of Mesh-1 where a poor correlation to the true solution is expected.

It is also interesting to point out that whilst the J-integral involves the integration of a strain energy term, the problems associated with the S-theory do not appear, as it has been shown by Eftis *et al* [7] that the truncation of the higher order terms has no effect on J and Eq. 3.10. Another feature of J is that the integration process tends to nullify the systematic scatter of the numerical data and was therefore able to predict K_I accurately without the application of any best fit technique. In practice, it is most likely that J is used in a non-linear analysis where stress intensity factors are not defined. Hence, J is not generally used for computing K_I, but instead, it is calculated and compared directly to a critical value J_c (which, like K_{I_c} and S_c, is a material parameter) to determine whether failure is to occur. The above results merely show that a finite elements approach to the J-integral can be applied with some confidence. However, it must be remembered that the restriction of self-similar growth must be observed, and that its usefulness when extended to true elasto-plastic materials is yet uncertain [17].

3.2 Predicting the Direction of Growth

Having seen how the neglected higher order term can affect the prediction of crack initiation by the S-theory, its effect on the prediction of the direction of crack extension is now examined. In testing the maximum normal tensile stress theory for predicting the direction of crack extension, Williams and Ewing [18] conducted experiments on PMMA (polymethylmethacrylate) sheet specimens which contain a central inclined slit crack and subjected to uniaxial tensile loads. It was found, particularly for steep crack angles ($\beta \to 0°$), that the angle of growth deviated from the predicted results presented by Erdogan and Sih [19]. The authors attributed the discrepancies to the fact that only a one-term expansion was used in the expression for the stresses, and showed that by selecting a critical parameter $\alpha = \sqrt{2r/a} = 0.1 (r/a = 0.005)$, and including higher order terms in the expansion, a better fit of experimental data with theory is achieved. In a subsequent discussion, Erdogan and Sih [20] showed that, unlike the maximum normal tensile stress criterion, the strain energy density criterion provides a better fit to the experimental data and is relatively insensitive to the parameter α. To investigate this point further, Eq. 2.12 is differentiated with respect to θ to form

$$\begin{aligned} \frac{\partial S}{\partial \theta} &= \frac{\partial A}{\partial \theta} + \frac{\partial B}{\partial \theta}\sqrt{r}, \\ \frac{\partial^2 S}{\partial \theta^2} &= \frac{\partial^2 A}{\partial \theta^2} + \frac{\partial^2 B}{\partial \theta^2}\sqrt{r}. \end{aligned} \tag{3.13}$$

and from Hypothesis 1 of the criterion, the direction of crack propagation is determined when

$$\frac{\partial S}{\partial \theta} = 0, \qquad and \qquad \frac{\partial^2 S}{\partial \theta^2} > 0. \tag{3.14}$$

Assuming $\nu = 0.33$ and for plane strain conditions, the solutions to Eq. 3.14 over the range $0° \le \beta \le 90°$ for $\alpha = 0, 0.1$ and 0.2 are presented in Fig. 8 together with the scatter band of the results in [18]. It may be seen that the one-term representation of the S-theory (equivalent to the case $\alpha = 0$) fitted the experimental data better than the one-term approach to the maximum stress criterion. It is also noted that the solutions for both $\alpha = 0.1$ and 0.2 fall within or near the experimental scatter band. However, there is some evidence that the use of $\alpha = 0$ for $\beta < 45°$ and $\alpha = 0.2$ for $\beta > 45°$ can result in a more accurate prediction. This has the implication that the core radius r_o may in fact not be a material property. The question of whether r_o may be considered as a material constant has previously been addressed by Chang [21]. Chang found that in order to obtain reasonable agreement between the experimental data of [18], [22], [23] and the strain energy density theory, various values of r_o/a (ranging from 0.005 to 0.15) had to be used despite the fact that all data were presented for the same material (PMMA) and for similarly sized cracks. He therefore concluded that r_o/a (and hence r_o) 'can hardly be justified as a material parameter in the S-theory'.

Chang [21] also discussed, at length, the dilemmas which may arise in applying the S-theory when no relative minimum, or alternately, when more than one relative minimum in S exist in the solution. Swedlow [24] showed that for uniaxial loading configurations, the choice of the global minimum leads to incorrect predictions. Swedlow then proposed the additional requirement that the S_{min} which governs fracture must be associated with a tensile hoop stress. On the other hand, Sih and Madenci [10] assert that it is the maximum of all S_{min}'s which first reaches the material threshold and is therefore the critical factor. For the inclined crack problem considered, two local minima are found for all β except for $\beta = 90°$, and indeed, it is the maximum of the two at any given r which corresponds to the results presented in Fig. 8. However, an interesting situation can arise when other loading conditions are considered. As an example, under a biaxial tension-compression loading system with $k = -1$, and for $\beta = 60°$, the paths of the two minima are as shown in Fig. 9. The corresponding plots of S along the paths denoted by i and ii are presented in Fig. 10. It is clear from the plots that the determination of the maximum of the minima would depend on the selection of r/a.

It has been seen in the uniaxial case that although the use of the S-theory to predict crack initiation requires an extremely small value of r/a, the restriction was much less severe for predicting the propagation angle as reasonable predictions may be achieved for the range $0 \leq r/a \leq 0.02$. However, the above example shows that this may not be valid in general as the choice of r/a appears to be crucial in determining whether the crack is to grow in direction i or ii. Perhaps this is also evidence for the possibility that the S-theory alone may be insufficient in determining the direction of crack propagation in general, and a modification such as that proposed by Swedlow [24] may be in order.

4. CONCLUSION

It has been shown that the usual assumption of a $1/r$ energy singularity is valid only within an extremely small regime around the crack tip. For the centre-cracked plate considered, this region is typically of the order of $r/a < 10^{-5}$. As a consequence, the application of the S-theory at some distance outside this region may result in substantial errors in the prediction of crack initiation. For a finite element analysis, an extrapolation technique using the more accurate form is proposed and found to be useful. However, this technique still requires a relatively fine computational grid, and a guideline for maximum mesh size, optimum number of data points and the valid extrapolation domain for arbitrary crack and loading configurations has yet to be established.

It has also been shown that the restriction on r/a is somewhat less severe for predicting the crack growth direction, and that for the uniaxial load case, reasonable agreement between experimental data and predicted results may be achieved over a relatively large range of r/a. Unfortunately, this may not be taken as a general rule as illustrated by the biaxial load example where a dilemma in choosing the correct S_{min} may arise when r/a is arbitrarily selected.

In closure, it should be emphasised that, despite the problems revealed by the current work, the strain energy density factor should not be disregarded as a useful parameter. There is no doubt that the S-theory works well under certain conditions, and its potential in handling mixed mode fractures is particularly valuable. The close relationship between S and the stress intensity factors (Eq. 2.12), and the reasonable agreement between predicted and existing experimental data on propagation angles, tend to support this. On the other hand, limitations to the theory must be identified and realised. Questions such as whether or not r_o is a valid material parameter, or what should be done when multiple minima in S exist, have, in the present author's opinion, yet to be positively resolved.

ACKNOWLEDGEMENTS

This work was carried out as part of a Commonwealth Advisory Aeronautical Research Council Co-operative project on fracture. The author wishes to thank Dr. Rhys Jones for his helpful comments.

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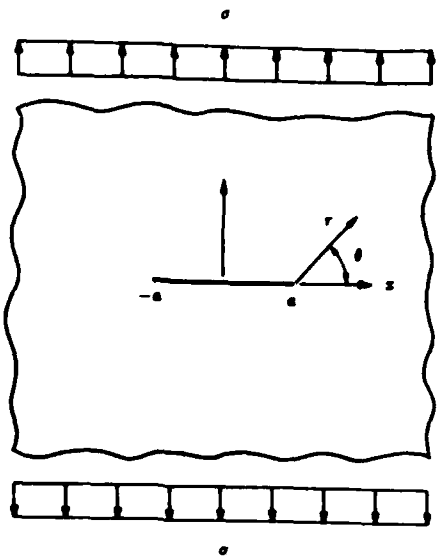

Figure 1. Uniaxially Loaded Flat-Crack Geometry

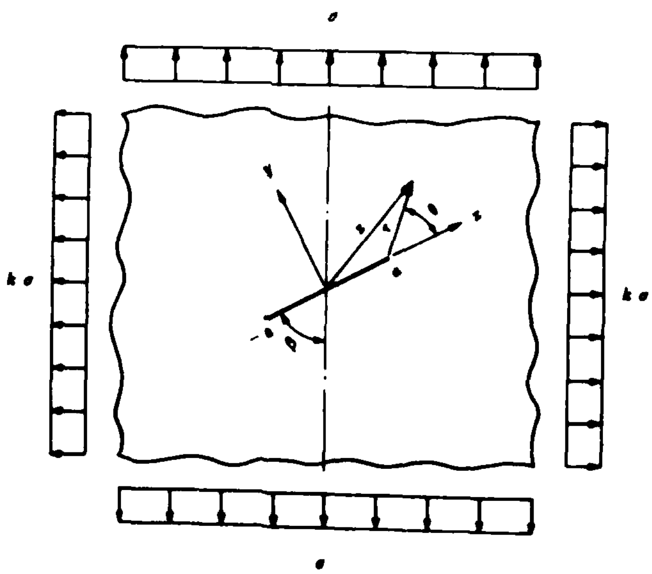

Figure 2. Biaxially Loaded Inclined-Crack Geometry

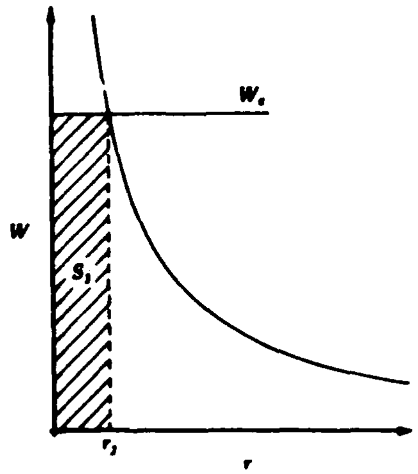

Figure 3. Strain Energy Density Distribution Along the Critical Direction

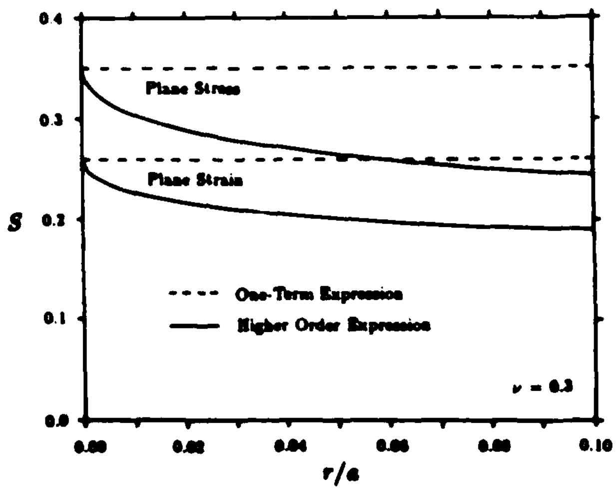

Figure 4. Comparison of the One-Term and the Higher-Order Representations of S

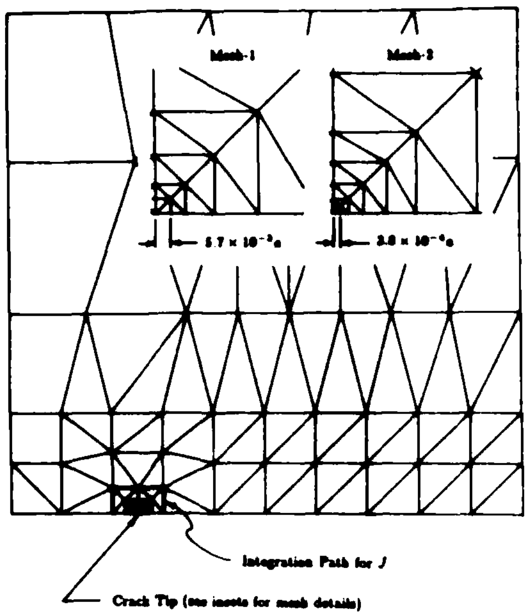

Figure 5. Finite Element Model of a Centre-Cracked Panel

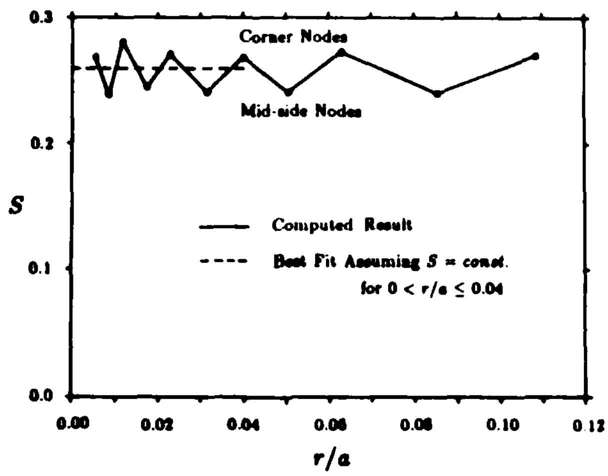

Figure 6. Computed S (Mesh-1)

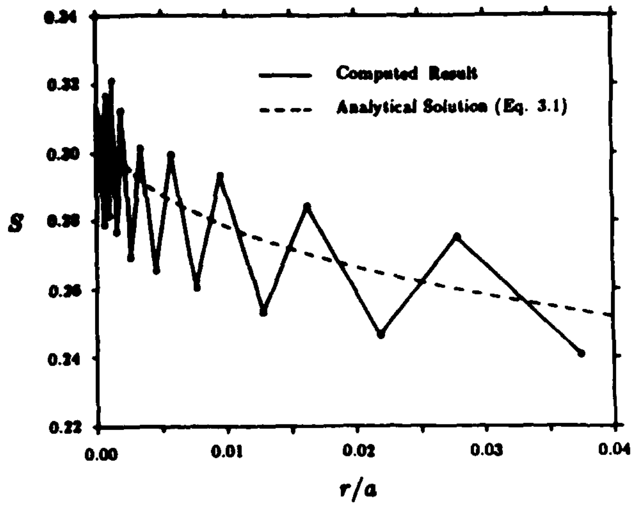

Figure 7. Computed S (Mesh-2)

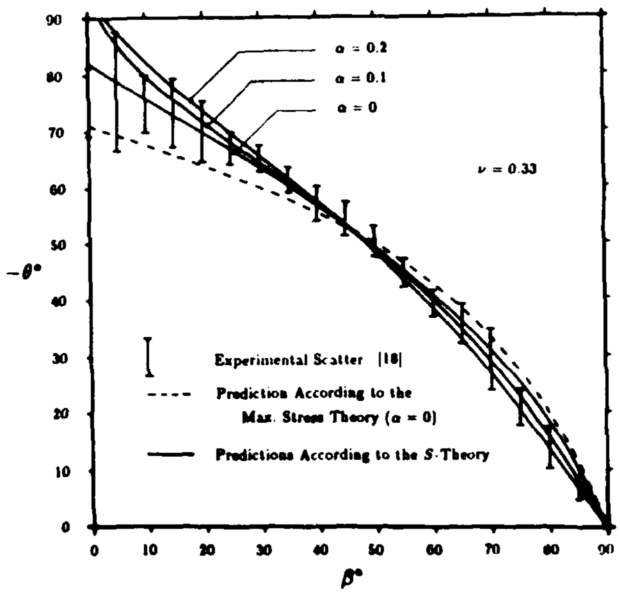

Figure 8. Prediction of Crack Extension Direction for the Inclined Crack Problem Under Uniaxial Loading

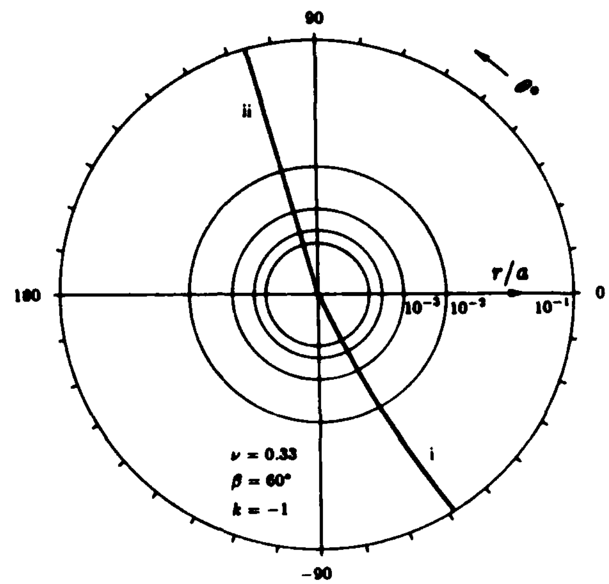

Figure 9. Paths of Minimum Strain Energy Density Factor for Biaxial Load Example

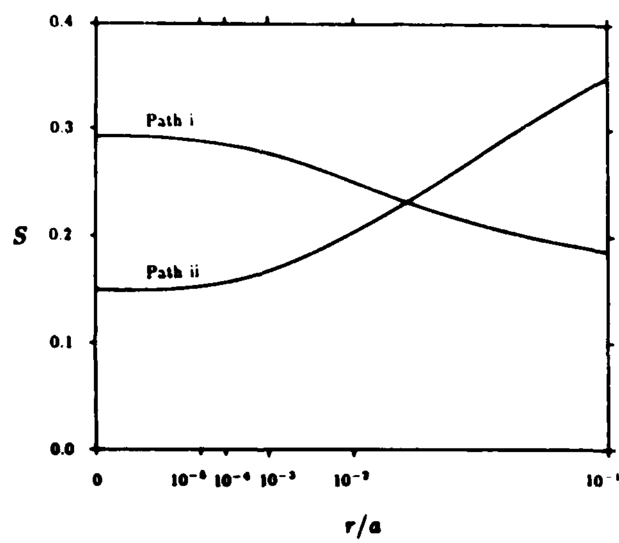

Figure 10. *S*-Distributions Along Paths i and ii of Fig. 9

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16. Abstract

Until recently, the one-parameter singular expression for stresses near a crack-tip was widely thought to be sufficiently accurate over a reasonable region for any geometry and loading conditions. This view has been fast changing due to the growing evidence that the inclusion of higher order terms can significantly affect the solution, particularly under certain biaxial loading conditions. In this context, the present paper examines the strain energy density criterion for fracture, and the consequences of the assumption of a 1/r energy singularity in the formulation on its application. It is found that this assumption imposes a rather severe restriction on the region for which the criterion is applicable, and that its application on an arbitrarily selected 'small' distance from the crack-tip (a procedure which has been adopted by many experimentalists and finite element analysts), can lead to erroneous results.

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17. Imprint Aeronautical Research Laboratories, Melbourne		
18. Document Series and Number Structures Report 421	19. Cost Code 211080	20. Type of Report and Period Covered
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